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Controlling complex networks with conformity behavior

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Abstract – Controlling complex networks accompanied by common conformity behavior is a fundamental problem in social and physical science. Conformity behavior that individuals tend to follow the majority in their neighborhood is common in human society and animal communities. Despite recent progress in understanding controllability of complex networks, the existent controllability theories cannot be directly applied to networks associated with conformity. Here we propose a simple model to incorporate conformity-based decision making into the evolution of a network system, which allows us to employ the exact controllability theory to explore the controllability of such systems. We offer rigorous theoretical results of controllability for representative regular networks. We also explore real networks in different fields and some typical model networks, finding some interesting results that are different from the predictions of structural and exact controllability theory in the absence of conformity. We finally present an example of steering a real social network to some target states to further validate our controllability theory and tools. Our work offers a more realistic understanding of network controllability with conformity behavior and can have potential applications in networked evolutionary games, opinion dynamics and many other complex networked systems.

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Introduction. – In the past four years, we have witnessed the rapid development of the controllability theory for complex networked systems [1–5]. Understanding our ability to control a variety of complex networks is a stepping stone towards achieving ultimate control of them, one of the key research goals in contemporary science [6–12]. A dynamical system is controllable if it can be driven from any initial state to any desired final state with infinite time by external controllers. We call the nodes controlled by external controllers driver nodes. Since the seminal work of Liu *et al.* [1], in which a general controllability framework for directed complex networks is offered, much effort has been devoted to applying the theoretical tools to investigate controllability of complex networks in many different fields, and to improving the controllability framework so as to broaden its application scope [13–19]. In particular, a more general controllability framework, named exact controllability of complex networks, has been proposed [2,15]. The tools offered by the improved framework can be applied to networks with any structural properties, including both directed and undirected networks with arbitrary link weights. The exact controllability theory can yield the same results as Liu's theory when both theories are applicable. More importantly, the exact controllability theory is based on eigenvalues and geometrical multiplicity of network matrix, which greatly facilitates the analysis in virtue of highly developed knowledge of network spectral properties in network science.

Despite the development of the controllability theories, there still exists a certain gap between the theoretical frameworks and real complex networked systems. Specifically, the controllability theories are applicable to the canonical linear time-invariant system $\dot{x} = Ax+Bu$, where vector x captures the states of nodes and state matrix A captures the interaction patterns among nodes. In general, A is the transpose of the network matrix, *i.e.*, the network

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structure. The simple model of networked systems in some scenarios may be not sufficient to characterize real systems. For example, intrinsic nodal dynamics that plays a significant role in the collective behavior of a complex system is omitted [20,21], rendering the findings based on the original controllability frameworks unrealistic to some systems. The problem has been fully addressed quite recently, but only for linear nodal dynamics [20]. Another example is that the controllability of some biological networks predicted by the theories is inconsistent with experimental findings, especially for protein and gene networks [22,23]. This conflict stems from the difference between the interaction patterns in theoretical models and that in real biological systems. To solve this problem, a modified model accompanied by a quite different controllability theory is introduced to better mimic real biological systems and explore their controllability theoretically [24].

In analogy with biological systems, similar problems hide in social networked systems because social interaction patterns are not incorporated into the state matrix A. A representative example is conformity behavior [25,26]. Matrix A merely reflects the relationship structure among individuals, but fully neglects conformity behavior during the evolution of a social network. Conformity behavior is quite common in society and animal communities. Some laboratorial experiments have revealed that in evolutionary games taking place in complex networks, people highly tend to follow the majority in their neighborhood rather than learn from some specific neighbors [27]. Human brain substrates of long-term memory conformity have been discovered in experiments as well [28]. In a group of animals, such as chimpanzees and monkeys, experiments have demonstrated that conformity behavior in terms of social learning facilitates the form of social and cultural norms [29,30]. Conformity behavior also plays a significant role in achieving consensus in a group of moving animals, such as fishes and birds [31–35]. Thus, it is imperative to incorporate conformity behavior into the controllability framework to better understand the controllability of social networked systems, which has not been tackled so far.

In this paper, we aim at exploring the controllability of complex networks with conformity behavior by relying on the exact controllability theory. We propose a simple and general network model to capture conformity behavior in decision-making processes of individuals, and show how to employ the exact controllability theory to identify the minimum number of driver nodes for fully steering the system to any target state. We explore the controllability that is defined by the ratio of the minimum number of drivers to the network size, of a variety of regular and complex networks with conformity, finding some interesting results that are different from the predictions of structural and exact controllability theory in the absence of conformity. We finally present an example of steering a small social network, in which evolutionary games take place, to several target states by controlling driver nodes with minimum number. Our work opens a new route to applying the controllability framework to real social networks in a more realistic manner, and our results are more practical than those obtained directly using the controllability framework without considering conformity behavior.

Model. – We consider nodes in the networks that are occupied by individuals. Initially, all individuals are allowed to choose their strategies in a continuous region $[0,1]$. In other words, the strategies of individuals are continuous with lower and upper limits. In evolutionary games, 0 and 1 denote complete defection and cooperation, respectively. In opinion spreading dynamics, 0 and 1 denote complete support of opinion 1 and 2, respectively. A value between 0 and 1 indicates the tendency of a player to choose 0 or 1. For memory-one strategy games in the presence of conformity behavior [36,37], the strategy of a player used in next round is determined by her/himself and her/his neighborhood together. For example, for a fair player, the probability to cooperate is determined by the fraction of cooperators among neighbors in the previous round: $x(t+1) = j(t)/n(t)$, where $j(t)$ and $n(t)$ are the number of cooperators and neighbors in t round, respectively [38]. The support degree of players to an opinion also tends to be consistent with other co-players in the neighborhood. These behaviors constitute nothing but conformity.

In the following, we take the evolutionary game as a representative example to formulate our controllability framework. Because of the conformity behavior, the probability to cooperate (strategy) for a player is proportional to the average cooperation probability among her/his co-players (neighbors) in the previous round:

$$x_i(t+1) = \sum_{j=1}^{n_i} x_j(t)/k_i, \tag{1}$$

where $x_j(t)$ is the strategy of node i's neighbor j, n_i is the number of neighbors of node i, $k_i = \sum_{j=1}^{N} A_{ij}$ is the degree of node i and A_{ij} is the connectivity matrix. Equation (1) indicates that player i makes decisions in the next round according to the average cooperation tendency in i's neighborhood in the current round, a typical conformity behavior. Particularly, if all of the neighbors of i are complete cooperators or defectors, i in the next round will choose to fully cooperate 0 or defect 1, respectively. Thus, the strategy evolution of the networked system can be formulated into the following linear equations:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_N(t+1) \end{bmatrix} = \begin{bmatrix} k_1^{-1} & 0 & \dots & 0 \\ 0 & k_2^{-1} & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & k_N^{-1} \end{bmatrix} A \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \tag{2}$$

or $x(t+1) = K^{-1}Ax(t)$, where we use K^{-1} (if $k_i = 0$, we set $k_i^{-1} = 0$) to denote the diagonal matrix consisting of

Table 1: Eigenvalues and minimum number of driver nodes of regular unweighted, undirected graphs with conformity behavior. $N_{\mathrm{D}}^{\mathrm{MMT}}$ denotes the minimum number of drivers calculated from the maximum geometric multiplicity. $q = 1, 2, \cdots, N$ and the geometric multiplicity of eigenvalues is indicated in "()" for fully connected networks. The details can be found in the appendix.

Network	Eigenvalue	$N_{\mathrm{D}}^{\mathrm{MMT}}$
Chain	$\cos\frac{(q-1)\pi}{N-1}$	1
Ring network	$\cos\frac{2\pi(q-1)}{N}$	2
Fully connected network	$-\frac{1}{N-1}(N-1), 1(1)$	$N-1$

the inverse of node degrees, and K^{-1} captures the conformity behavior.

For the discrete time system (2), the underlying framework of controllability is the same as for continuous time systems. Therefore, the controllability of system (2) can be investigated by using the exact controllability theory [2]. Specifically, because matrix $K^{-1}A$ is asymmetric, the minimum number N_{D} of driver nodes is determined by the maximum geometric multiplicity:

$$N_{\mathrm{D}} = \max_i\{\mu(\lambda_i)\}, \tag{3}$$

where $\mu(\lambda_i) = N - \mathrm{rank}(\lambda_i I_N - K^{-1}A)$ is the geometric multiplicity of distinct eigenvalues λ_i of the matrix $K^{-1}A$. If network $K^{-1}A$ is sparse, according to the efficient formula of the exact controllability theory, we have

$$N_{\mathrm{D}} = \max\{1, N - \mathrm{rank}(K^{-1}A)\}. \tag{4}$$

For a dense network with identical link weights w, N_{D} is given by

$$N_{\mathrm{D}} = \max\{1, N - \mathrm{rank}(wI_N + K^{-1}A)\}. \tag{5}$$

The efficient formula allows us to compute N_{D} in a much more efficient manner. The controllability of a network is defined by the ratio of the number N_{D} of driver nodes to the network size N: $n_{\mathrm{D}} = N_{\mathrm{D}}/N$.

According to the exact controllability theory [2], we implement the elementary column transformation to identify all linear correlations in matrix $\lambda^{\mathrm{M}} I_N - K^{-1}A$, which is able to yield all driver nodes located via control matrix B to ensure

$$\mathrm{rank}[\lambda^{\mathrm{M}} I_N - K^{-1}A, B] = N, \tag{6}$$

where λ^{M} is the eigenvalue corresponding to the maximum geometric multiplicity.

Results. – We first explore the controllability of chain, ring networks and other regular networks with conformity behavior. For regular networks, their eigenvalues can be calculated precisely (see appendix for details), accounting for rigorous theoretical predictions of N_{D} based on the exact controllability theory. Table 1 shows theoretically the eigenvalues corresponding to the maximum multiplicity and N_{D} of three undirected regular networks. We see that the eigenvalues are different from those without conformity behavior [2], but N_{D} are the same as that in the absence of conformity. Moreover, we can see that chain and ring networks are much easier to be controlled than fully connected networks with conformity behavior. Nearly all nodes need to be controlled to ensure fully control of a fully connected network. This finding is quite different from the prediction of the structural controllability theory that from the structural point of view without conformity, a single driver node is sufficient to fully control a fully connected network. For a network with conformity behavior, N_{D} is determined by both coupling matrix A and matrix K^{-1}. Insofar as the network is fully connected, we can simply derive N_{D} nodes based on the efficient formula given in eq. (5):

$$\begin{aligned} N_{\mathrm{D}} &= \max\{1, N - \mathrm{rank}(wI_N + K^{-1}A)\} \\ &= \max\{1, N - \mathrm{rank}(wE_N)\} = N - 1, \end{aligned} \tag{7}$$

where $w = 1/(N-1)$ and E_N is a matrix in which all elements are 1.

Table 2: Exact controllability measures of real undirected and directed networks with conformity behavior. For each network, we show its type, name and density of driver nodes calculated in the real network. n_{D} and $n_{\text{non-D}}$ denote the minimum fraction of drivers calculated from the maximum geometric multiplicity of eq. (3) including conformity and without conformity, respectively. D and U denote directed network and undirected network, respectively. For data sources and references, see Supplementary Information of ref. [2].

Type	Name	Class	n_{D}	$n_{\text{non-D}}$
Food web	Grassland	D	0.5227	0.5227
	Little Rock	D	0.7541	0.7541
	Seagrass	D	0.3265	0.3265
	Silwood park	D	0.7662	0.7662
	St. Martin Island	D	0.4000	0.4000
	Ythan	D	0.5185	0.5185
Electronic circuits	s208a	D	0.2377	0.2377
	s420a	D	0.2341	0.2341
	s838a	D	0.2324	0.2324
Neuronal	Celegans	D	0.1650	0.1650
Trust	Prison inmate	D	0.1343	0.1343
	WikiVote	D	0.6656	0.6656
Social network	Dolphins	U	0.0323	0.0323
	Football	U	0.0087	0.0087
	karate	U	0.2941	0.2941
	Polbooks	U	0.0095	0.0095
Metabolic	*C. elegans*	U	0.3245	0.3245
Transportation	USA top-500 Airport	U	0.2500	0.2500

Table 2 shows the exact controllability measure of some real-undirected and -directed networks. Overall, the biological networks need a larger fraction of driver nodes, whereas the social networks only need a smaller fraction of driver nodes to achieve full control. Moreover, the undirected networks are easier to control than directed networks. Note that here the controllability is determined by matrix $K^{-1}A$. The matrix, because of K^{-1}, becomes asymmetric, fundamentally different from the symmetric

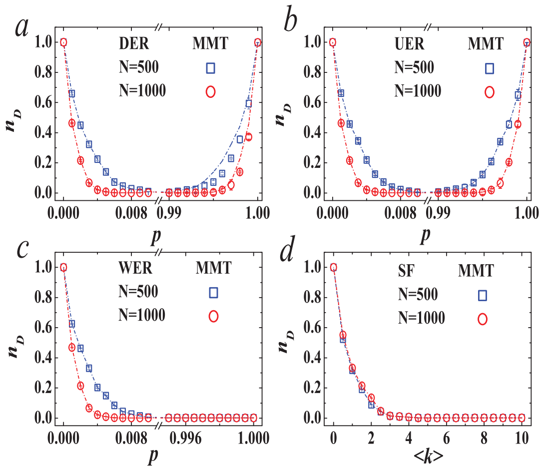

Fig. 1: (Color online) Exact controllability of networks without conformity (dashed line) and including conformity (symbols). Exact controllability measure n_{D} as a function of the connecting probability p for (a) directed and unweighted ER random networks (DER); (b) undirected and unweighted ER random networks (UER); (c) undirected ER random networks with random link weights (WER); (d) n_{D} *vs.* the average degree for undirected and unweigthed scale-free networks with power-law exponent 3. Each data point is obtained by calculation of maximum geometric multiplicity of eq. (3) and an average of 20 independent realizations. The error bars denote the standard deviation and N is the network size.

network matrix A. Thus, n_{D} of networks with conformity supposes to be different from that as determined by A without conformity. However, we find that n_{D} of the empirical networks with conformity is exactly the same as that in the absence of conformity. This result stems from the sparsity of most real networks. In general, based on the results in ref. [2], link weights do not play a significant role in n_{D} in sparse networks. For matrix $K^{-1}A$, its main difference from matrix A lies in the element values in $K^{-1}A$ rather than the positions of nonzero elements, accounting for similar controllability to the network characterized exclusively by A without conformity, as displayed in table 1.

Figure 1(a) shows n_{D} as a function of the connecting probability p for directed and unweighted Erdős-Rényi (ER) random networks. We find that, for sparse networks, the fraction of driver nodes n_{D} decreases monotonically with connecting probability p. This indicates that controllability is promoted by higher density of connections for sparse networks. By contrast, for sufficiently large p, n_{D} increases as p increases because of the impact of identical link weights. Eventually, when p approaches 1, we nearly need to control all nodes, which is similar to the result of the fully connected network in table 1. This finding holds for different network sizes. Especially, by comparing n_{D} without conformity (dashed line) and including conformity (symbols), we find that, for directed dense ER networks, conformity behavior can promote the controllability, while fig. 1(b) shows that, for undirected and unweighted random networks, n_{D} without conformity and including conformity for undirected networks are exactly the same. For dense networks, two nodes i and j generally have the same out-neighbors. If the elements w_{ij} and w_{ji} in matrix $\lambda_i I_N - K^{-1}A$ also meet the condition $\lambda_i = -w_{ij} = -w_{ji}$ (w is the weight caused by conformity), then, the i-th and j-th columns are linearly dependent, and we need to control one of the nodes. For undirected networks, in-degree equals out-degree, and we surely have $w_{ij} = k_{in}(i)^{-1} = w_{ji} = k_{in}(j)^{-1}$ (for sparse networks, $w_{ij} = w_{ji} = 0, 1$). However for directed networks, two nodes which have the same out-neighbors may have distinct in-degrees, which causes $w_{ij} \neq w_{ij}$. At this point, two columns are linearly independent, and then n_{D} of the network including conformity is smaller than that without conformity. Figure 1(c) shows n_{D} for undirected random networks with random link weights. We see that n_{D} decreases monotonically with connecting probability p. In contrast to the result in fig. 1(a) and (b), due to random link weights, n_{D} no longer increases with connecting probability p for dense networks. Figure 1(d) shows n_{D} as a function of the average degree $\langle k\rangle$ for unweighted scale-free networks (scale-free networks are constructed by using static model [39]). We find that, n_{D} decreases rapidly to $1/N$ as $\langle k\rangle$ increases. n_{D} of unweighted scale-free networks and scale-free with random link weights are nearly the same, indicating the effect of link weights is small. Figure 1 demonstrates that, on the one hand, for sparse networks, higher densities of links favor controllability. On the other hand, for dense networks with identical link weights, higher densities of links render controllability weaker.

To further validate our controllability methods, we use our tools to control a real small social network to some target states. Figure 2(a) shows the network constituted by 22 participants from Arizona State University [40]. There is a link between two individuals if they are friends with each other. Figure 2(b) shows the evolution of strategies in the absence of any external inputs. We find that, the strategies of nodes in the connected component rapidly achieve a homogeneous stable state, determined by both the initial strategy distribution and the network structure. We use our method to identify driver nodes (nodes in purple in fig. 2(a)) and impose external control signals to control the strategies of individuals to some desired states. Identified driver nodes include two isolated nodes and two nodes of degree 2. Under our control, three desired final states are achieved from the same initial strategy distribution within 300 rounds, as shown fig. 2(b)–(e), providing strong support for the applicability of our control method to networks with conformity effect.

Conclusion. – We have proposed a simple model to characterize conformity dynamics taking place in complex networks and demonstrated how to control the networks by applying the exact controllability theory, including identifying driver nodes with minimum number of a variety of regular, model and empirical networks, and

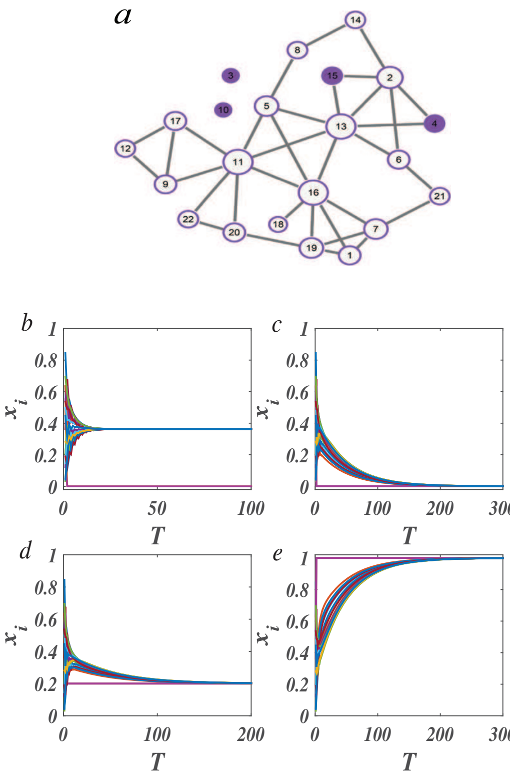

Fig. 2: (Color online) Structure and control of a social network. (a) Structure of the social network. The driver nodes are filled circles, including two isolated nodes 3, 10 and nodes 4, 15 whose degree $k = 2$. The size of a node is proportional to its degree. (b) Time evolving of nodes' strategies without external control inputs. Time evolving of nodes' strategies by controlling the driver nodes and final states are set as (c) $x_i = 0$, (d) $x_i = 0.2$ and (e) $x_i = 1$ for all of the nodes.

implementing an actual control of a small social network. Due to the conformity effect, the original symmetric network matrix is converted into an asymmetric matrix representation, which supposes to render controllability quite different. We have found that the difference is only obvious for directed dense networks. That is, conformity can promote the controllability. However, for other networks, this difference is actually small. For sparse networks, the small difference is ascribed to the small influence of link weights, although they induce the matrix asymmetry. For very dense undirected networks with identical link weights, the asymmetric property becomes weak, yielding approximately a symmetric matrix with identical weights. Thus, the controllability of very dense undirected networks with identical link weights and with conformity effect is quite close to that without conformity. Nevertheless, incorporating the conformity effect into social networks will offer a more exact and deeper understanding of our ability to control such networks in terms of steering a minimum number of driver nodes. Our work also sheds light into how to use controllability theories to explore complex networks in a more realistic manner rather than directly and carelessly use the theories.

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Appendix: controllability of regular graphs. – *Undirected chain graph.* Matrix $K^{-1}A$ for an undirected chain with N nodes is [2,41,42]

$$K^{-1}A = \begin{bmatrix} 0 & 1 & & & \\ 0.5 & 0 & 0.5 & & \\ & \ddots & \ddots & \ddots & \\ & & 0.5 & 0 & 0.5 \\ & & & 0 & 1 & 0 \end{bmatrix}, \tag{A.1}$$

where all unwritten elements are zeros. The eigenvector α of eigenvalue λ satisfies $(\lambda I_N - K^{-1}A)\alpha = 0$, which can be written for each component as

$$\begin{cases} \lambda\alpha_1 - \alpha_2 = 0, \\ -0.5\alpha_{k-1} + \lambda\alpha_k - 0.5\alpha_{k+1} = 0 \qquad (2 \le k \le N-1), \\ -\alpha_{N-1} + \lambda\alpha_N = 0. \end{cases} \tag{A.2}$$

The boundary requirement is

$$\begin{cases} \alpha_0 = \lambda\alpha_1 = \alpha_2, \\ \alpha_{N+1} = \lambda\alpha_N = \alpha_{N-1}. \end{cases} \tag{A.3}$$

Then eq. (A.2) can be written as

$$-\alpha_{k-1} + 2\lambda\alpha_k - \alpha_{k+1} = 0 \quad (0 \le k \le N-1). \tag{A.4}$$

The general solution is $\alpha_k = ar_1^k + br_2^k$, where r_1 and r_2 are the roots of the corresponding polynomial $x^2 - 2\lambda + 1 = 0$, which satisfy

$$\begin{cases} r_1 + r_2 = 2\lambda, \\ r_1 r_2 = 1. \end{cases} \tag{A.5}$$

Substituting the general solution into boundary requirement, we obtain

$$\begin{cases} \dfrac{a}{b} = -\dfrac{1 - r_2^2}{1 - r_1^2}, \\ \dfrac{a}{b} = -\dfrac{1 - r_2^2}{1 - r_1^2}\dfrac{r_2^{N-1}}{r_1^{N-1}}. \end{cases} \tag{A.6}$$

Thus, $(\frac{r_2}{r_1})^{N-1} = 1$ or $\frac{r_2}{r_1} = e^{\frac{2\pi q\sqrt{-1}}{N-1}}$ $(q = 0, 1, \cdots, N-1)$ or $\frac{r_2}{r_1} = e^{\frac{2\pi(q-1)\sqrt{-1}}{N-1}}$ $(q = 1, 2, \cdots, N)$. Substituting $r_2 = r_1 e^{\frac{2\pi(q-1)\sqrt{-1}}{N-1}}$ into the last equation of (A.6) yields

$$\begin{cases} r_1 = e^{\frac{\pi(q-1)\sqrt{-1}}{N-1}}, \\ r_2 = e^{-\frac{\pi(q-1)\sqrt{-1}}{N-1}}. \end{cases} \tag{A.7}$$

Substituting eq. (A.7) into the first equation of (A.5), we have

$$\lambda = \frac{1}{2}(e^{\frac{\pi(q-1)\sqrt{-1}}{N-1}} + e^{-\frac{\pi(q-1)\sqrt{-1}}{N-1}}) = \cos\frac{(q-1)\pi}{N-1}. \quad (A.8)$$

To calculate the maximum geometric multiplicity of all distinct eigenvalues, we need to obtain the rank of matrix $(\lambda I_N - K^{-1}A)$:

$$\lambda I_N - K^{-1}A = \begin{bmatrix} \lambda & -1 & & & \\ -0.5 & \lambda & -0.5 & & \\ & \ddots & \ddots & \ddots & \\ & -0.5 & \lambda & -0.5 & \\ & 0 & -1 & \lambda & \end{bmatrix}. \quad (A.9)$$

For each eigenvalue of $K^{-1}A$, the determinant of matrix $\det(\lambda I_N - K^{-1}A) = 0$. But the eigenvalues of $K^{-1}A$ are not the eigenvalues of the cofactor matrix of $K^{-1}A$. Hence, $\det(D_{n-1}[\lambda I_N - K^{-1}A]) \neq 0$. So, rank $[\lambda I_N - K^{-1}A] = N-1$ for all eigenvalues and $\mu(\lambda_i) = 1$.

Ring network and fully connected network. For the ring network and the fully connected network, all nodes have the same degree. Then, matrix $K^{-1}A$ equals the identity matrix times the reciprocal of the degree. The eigenvalues of the matrix $K^{-1}A$ are the same as the eigenvalues of the matrix A multiplied by the reciprocal of the degree. According to refs. [2,41,42], we can obtain the eigenvalues of $K^{-1}A$ for the ring network are $\lambda_q = \cos\frac{2(q-1)\pi}{N}$ with $\lambda_q = \lambda_{N-q+2}$ and $\delta(\lambda_q) = 2(q = 1, 2, \cdots, N)$ for $N > 4$. For undirected networks, the maximum geometric multiplicity equals the maximum algebraic multiplicity. Hence, $\mu(\lambda_q) = 2$.

The eigenvalues for the fully connected network and the respective algebraic multiplicities are $\lambda_1 = 1$, $\delta(\lambda_1) = 1$ and $\lambda_2 = -\frac{1}{N-1}$, $\delta(\lambda_2) = N-1$ and $\mu(\lambda_q) = N-1$.

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